ALMA Memo 391
A Square Wave Phase-Switching Scheme for a Large Number of Antennas.

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Abstract

The complete set of square waves with frequencies that are integral multiples of a unit frequency cannot be used for phase switching since such a set are not mutually orthogonal, as discussed in ALMA Memos 287 and 385. However, by recognizing that absolute mathematical orthogonality is not required, and by selecting a sub-set of square-waves of certain frequencies, it is possible to design a practical phase switching scheme. Compared to the use of Walsh functions, this scheme has the advantage that the values of cross correlation between pairs of square waves remain very small in the presence of arbitrary relative time shifts.

Square waves of Rademacher sets (frequencies proportional to 2^k where k is a non-negative integer) are mutually orthogonal and can be used for phase switching. Since, however, the range of frequencies required increases exponentially with the number of antennas, Rademacher sets are not useful when the number of antennas is large. The possibility of using square waves of frequencies that are integral multiples of a fundamental frequency has been discussed in ALMA Memos 287 and 385 (D'Addario 2000, 2001), but deemed impractical because square waves of such a set are not mutually orthogonal. However, it was noted in ALMA Memo 287 that to simplify the generation of the waveforms a cross correlation low enough to suppress unwanted effects by 10^{-4}, rather than strict orthogonality, would be satisfactory. Here we show that by using the same cross correlation requirement for selection of the square waves, it is possible to obtain sets that are practical for phase switching with much larger numbers of antennas than is possible with Rademacher sets.

A square wave of frequency m times a fundamental frequency f_0 contains nonzero Fourier components of frequency $(2i+1)mf_0$, where i and m are non-negative integers. Square waves of frequency mf_0 and nf_0 will have common non-zero Fourier components given by solutions of the equation

$$(2i+1)mf_0 = (2j+1)nf_0, \tag{1}$$

where j and n are also non-negative integers. Since a natural number can be factored into primes in one and only one way, both sides of Eq. (1) must have the same prime factors. If n and m are prime numbers, then $(2i+1)$ must contain n as a factor, and $(2j+1)$ must contain m. Thus we have

$$(2i+1) = (2p+1)n \quad \text{and} \quad (2j+1) = (2q+1)m, \tag{2}$$

for non-negative integers p and q. Then from Eq. (1)

$$(2p+1)nm = (2q+1)nm. \tag{3}$$

Hence $p = q$ and common Fourier components exist with frequencies $(2p+1)nmf_0$. For any such component of square wave mf_0 the harmonic number is $(2p+1)n$, and for square wave nf_0 the harmonic number of the component with the same frequency is $(2p+1)m$. Consider square waves of unit amplitude (i.e. switching between ± 1). Then the amplitudes of the Fourier components of a square wave are equal to $4/\pi$ times the reciprocal of the harmonic number. The product of the harmonic numbers of two components of frequency $(2p+1)nmf_0$ is $(2p+1)^2nm$. If the two components are in phase, the integral of their product over one period of the fundamental (which we take to be equal to the minimum data-integration time) is

$$\frac{4^2}{\pi^2(2p+1)^2nm}\int_0^{1/f_0}\sin^2[2\pi(2p+1)nmf_0t]dt = \frac{8}{\pi^2(2p+1)^2nmf_0}. \tag{4}$$

The product of two square waves integrated over a period $1/f_0$, which we shall refer to as the cross correlation, can be obtained by summing the pair-products of the common Fourier components, as follows:

$$\frac{8}{\pi^2nmf_0}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2} = \frac{1}{nmf_0} \tag{5}$$

Here we have assumed that the two Fourier components of each common frequency are aligned in phase. Relative time shifts between square waves can only reduce this cross correlation. Equation (5) represents the suppression factor for the relevant unwanted responses. We need to express it in relation to the response to the wanted signal, for which the corresponding factor is equal to the square of the amplitude of the square-waves integrated over the data averaging time, that is, $1/f_0$. Thus the suppression factor relative to the response to the wanted signal is $1/mn$. It follows that if m and n are prime numbers greater than 100, the suppression factor is always less than 10^{-4}. For example, for phase switching an array of 64 antennas, square waves with frequencies proportional to the prime numbers from 101 to 461 could be used. The Fourier components of these square waves are all odd multiples of f_0. The range of numbers can therefore be slightly reduced by including the Rademacher series of square waves with frequencies 2^kf_0, but excluding the fundamental f_0, since for these the Fourier components are even multiples of f_0. Then for 64 antennas the eight Rademacher functions with frequencies proportional to 2 to 256, plus the 56 square waves with frequencies proportional to the primes from 101 to 419, would suffice. As explained in ALMA Memo 287, to avoid introducing an error greater than 10^{-4} into the cross correlation, the square waves must be generated with sufficient accuracy that the relative errors in the intervals between transitions should be no greater than 10^{-4}. This requires using a clock frequency no less than 10^4 times the frequency of the fastest square wave, that is, 4.19 MHz for 1 s data-averaging time or 262 MHz for 16 ms. (Note that in generating the square waves in this way, each transition should be made at the clock pulse nearest the ideal transition time, which can be achieved by using a direct digital synthesizer.) For 64 antennas, use of Rademacher functions alone would not be practicable, since the range of frequencies required would be from 1 to $2^{63} \approx 10^{19}$. With Walsh functions, for 64 antennas the shortest interval between transitions is 1/64 of the averaging time if only the Cal functions, or only the Sal functions, are used. If both are used the minimum interval between Cal function transitions and the corresponding Sal function transitions is also 1/64 of the averaging time (although the minimum is 1/32 for the transitions in the same Walsh function). Thus, for the square wave scheme described above, the transitions are faster than required with Walsh functions by a factor of approximately 6.5, for 64 antennas. However, the square waves offer the advantage that arbitrary time delays between components can be tolerated.

In the ALMA system the signals will be digitized at the antennas. The phase switching will most probably be inserted on the first LO and removed by reversing the sign bit immediately after sampling. These two operations take place within the same electronics room of each antenna, and there should be no problem in keeping their relative timing good to a few tens of ns. Spurious signals that infiltrate the IFs will be phase switched at the sampler output, and if such signals are correlated between antennas their removal depends upon the degree of correlation of the phase switch waveforms when the signals arrive at the correlator. Before correlation some signals will be delayed to compensate for geometric delays, and for a baseline of, say, 10 km such a delay could amount to 33 μs. The timing accuracy required to maintain the achievable suppression with two Walsh functions at a level of approximately 10^{-4} depends upon the sequency of the Walsh functions. For example, in the case of the Cal and Sal functions of sequency 16, for which the Walsh functions are square waves, the suppression factor for the pair goes from zero to unity in a range of relative time shift of 1/4 of the square wave period, that is, 1/64 of the data-averaging time. Thus to maintain suppression of 10^{-4} would require a timing accuracy of $\sim$1.5 μs for a 1 s averaging time, or 25 ns for 16 ms averaging. So compensation for varying time delays would be required with Walsh functions, but not with the square wave scheme outlined above. (Note that in the VLA this timing problem is neatly avoided by placing all the transitions in the non-receiving intervals of the waveguide cycle.) For ALMA, the use of additional quadrature phase switching to separate sidebands of double-sideband front ends, if required, would mandate synchronization of the square waves at the correlator input and thereby fail to take advantage of the shift invariance. Note that the phase switching which suppresses components that infiltrate the IFs is introduced by sign reversal after digitization, and thus represents a fully accurate shift of π radians in phase. Thus the suppression is not degraded by errors in the phase-shift angle, although such errors may occur in the analog phase shift introduced in the local oscillator.

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References

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