ALMA Memo 385:

Orthogonal Functions For Phase Switching and a Correction to ALMA Memo 287

Larry R. D'Addario

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Introduction

In a synthesis array, it is possible to apply certain types of modulation to the signals from the various antenna elements and then, at a later point in the signal processing, to remove the effect of this modulation from the desired signal in such a way that some undesired components of the signal retain the modulation. If the modulation is different for each element and if this makes the undesired components mutually orthogonal over some time interval, then those components will contribute zero to the correlator output when the integrating time is a multiple of the orthogonality interval. A similar method can be used to separate two different desired components in the signal (e.g., upper and lower sidebands of a mixer), so that each can be measured separately. Often the modulation involves changing the phase of the signal among a small number of values, in which case it is called "phase switching." These ideas are discussed in more detail in [1], and their application to the ALMA telescope is discussed in [2].

Usually the modulation is multiplicative, in which case the modulating functions must form a mutually orthogonal set. For N elements we need a set of N such functions, and a key to implementing this scheme is finding an appropriate set. Especially useful are two-valued or binary functions. When these are synthesized from a common clock, so that a change in state can occur only on a clock "tick," then it is well known that the shortest mutually orthogonal set consists of Walsh functions [3], and that the minimum length is N ticks. Another useful set is that of Rademacher functions, which are square waves whose periods are $2^{k+1}t_0$ for distinct positive integers k, where t_0 is the tick interval. However, to obtain N distinct Rademacher functions requires that the orthogonality interval (longest period square wave) grow exponentially with N, resulting in impractical implementations except at small N. The Rademacher functions do have the advantage of retaining their orthogonality when offset in time relative to each other (*i.e.*, they need not be in phase); the Walsh functions do not have this property.

Analysis and Correction

It was claimed in [2] that another set of square waves, those with periods $(k+1)t_0$ for distinct positive integers k, is also mutually orthogonal over the period of the longest one, but that such functions are difficult to synthesize exactly. Unfortunately, this claim is incorrect. Functions of this set are generally not orthogonal over any time interval, as is easily seen by considering those of period 1 and 3 over an interval of length 3. Usually it is possible to find a time offset (phase) at which one pair is orthogonal, but this does not allow mutual orthogonality for $N > 2$. Here I give an intuitive explanation of which square waves are orthogonal, and then I use it to show that the Rademacher functions are the *only* mutually-orthogonal set of square waves under an arbitrary time shift.

I am grateful to Darrel Emerson (private communication) for the following explanation. For two co-periodic signals to have non-zero correlation, there must be some terms in their Fourier series where both have non-zero coefficients. An important property of square waves is that all even-numbered Fourier terms have zero coefficients and all odd-numbered terms have non-zero coefficients. It follows that, for example, square waves of frequencies 1 and 2 have no common Fourier terms with non-zero coefficients; but 1 and 3 have frequencies $3, 9, 15, \ldots$ in common; and 3 and 5 have $15, 45, 75, \ldots$ in common. Therefore, square waves of frequencies 1 and 2 are orthogonal, but neither 1 and 3 nor 3 and 5 can be orthogonal for arbitrary time shift.

We can generalize this by realizing that the non-zero Fourier terms of a square wave at frequency f_0 have frequencies of the form $(2i+1)f_0$ for integer $i \geq 0$, and for another square wave at a multiple of f_0, say nf_0, the non-zero terms are at $(2j+1)nf_0 = (2nj+n)f_0$ for integer $j \geq 0$. It follows that the two remain orthogonal if and only if n is even. The lowest-frequency such multiple is the one at $n = 2$. Now consider this last square wave at $2f_0 \equiv f_1$. By the same reasoning, the lowest frequency multiple orthogonal to it is at $2f_1 = 4f_0 \equiv f_2$. (We need not consider square wave frequencies other than multiples of f_1 since these are already known not to be orthogonal to f_0.) Continuing in the same way, we obtain at each step a square wave of frequency $f_k = 2^k f_0$ which is orthogonal to all those previously generated, and where no other multiple of f_0 less than f_k is orthogonal to all of them. We conclude that the lowest-frequency set of N orthogonal square waves consists of those at frequencies $\{2^k f_0, k = 0...N-1\}$, which are the Rademacher functions.

Incidentally, sinusoids of frequencies $0, 1, 2, 3, ...$ *do* form a mutually orthogonal set over the unit interval for arbitrary phases. Being continuous-valued functions, they are not as convenient for our purpose as binary ones if the demodulation is to be done digitally. But sinusoidal phase modulation (equivalent to a frequency offset) can be implemented in local oscillators, and ALMA will use this method for suppression of an unwanted sideband of DSB receivers [2].

Conclusion

Phase switching and other binary modulation schemes that rely on orthogonality must use Walsh functions or Rademacher functions, since no other mutually orthogonal binary functions are known. (For discrete functions with more than two states, a generalization of Walsh functions called shifted m-sequences [4] can be used.) For Walsh functions, the orthogonality depends on their being aligned in time at the point where the signals are combined (the cross-multipliers of the correlator in a synthesis telescope). For Rademacher functions such alignment is not necessary, but large sets of Rademacher functions span an inconvenient range of frequency or time.

References

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[3] H. F. Harmuth, 1972, *Transmission of Information by Orthogonal Functions*, 2nd ed., Berlin:Springer-Verlag.

[4] E. Keto, 1999, "Shifted m-sequences for phase switching." Submillimeter Array Project Tech. Memo No. 134, Smithsonian Astrophysical Observatory, 1999-Mar-24.